SOLUTIONS OF EXERCISES.

54

SHOW that if in a plane triangle ABC

$$\cos A + \cos B + \cos C = \sqrt{2},$$

the centre of the circumscribed circle lies on the circumference of the inscribed circle. [*R. D. Bohannan.*]

SOLUTION.

The above equation may be written

$$1 + 4 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C = \sqrt{2}\,;$$

whence, squaring,

$$8 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C + 16 \sin^2 \tfrac{1}{2}A \sin^2 \tfrac{1}{2}B \sin^2 \tfrac{1}{2}C - 1 = 0.$$

Dividing by $16 \cos^2 \tfrac{1}{2}A \cos^2 \tfrac{1}{2}B \cos^2 \tfrac{1}{2}C$, we have

$$\frac{\tan \tfrac{1}{2}A \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C}{2 \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C} + \tan^2 \tfrac{1}{2}A \tan^2 \tfrac{1}{2}B \tan^2 \tfrac{1}{2}C - \frac{1}{16 \cos^2 \tfrac{1}{2}A \cos^2 \tfrac{1}{2}B \cos^2 \tfrac{1}{2}C} = 0.$$

Reducing this equation by means of the relations existing between R, r, and the angles of a triangle, we have

$$2Rr + r^2 - R^2 = 0,$$

or

$$R^2 - 2Rr = r^2.$$

But $\sqrt{(R^2 - 2Rr)}$ equals the distance between the centres of the circumscribed and the inscribed circles; hence the proposition. [*Claude Waller.*]

74

FOUR points are taken at random on the surface of a sphere. What is the probability, that all of the points do not lie in the same hemisphere?

[*A. Hall.*]

SOLUTION.

Let three points be taken at random on the surface of a sphere; they will all lie in the same hemisphere, and will form the vertices of a spherical triangle. If right lines be drawn from these points to the centre of the sphere, and extended to the surface, the points of intersection will be the vertices of a triangle opposite the first one. If the fourth random point is not in the same hemisphere as the first three points, it will be on the surface of the opposite triangle. Now the average area of this triangle is $\tfrac{1}{2}\pi r^2$, or one-eighth the surface of the sphere. Hence the required probability is $\tfrac{1}{8}$. [*A. Hall.*]

80

In exercise 4, what is the probability that the random circle exceeds the average circle? [*Artemas Martin.*]

SOLUTION.

Let x be the radius of the random circle; then from equation (1), solution of exercise 4, Vol. I. p. 67, we have

$$y = \sqrt{(2rx)}. \qquad (1)$$

Let R be the radius of the average circle; then, p. 68,

$$\pi R^2 = \tfrac{1}{32}\pi r^2\left[1 - \frac{\sqrt{3}}{\sqrt{3} + 2\log(\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{6})}\right];$$

whence
$$R = \tfrac{1}{4}r\left[\tfrac{1}{2} - \frac{\sqrt{3}}{2\sqrt{3} + 4\log(\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{6})}\right]^{\frac{1}{2}}. \qquad (2)$$

If x be greater than R, the random circle will exceed the average circle. Putting R for x in (1) we get the inferior limit of y. The superior limit is $\frac{1}{2}r\sqrt{2}$, the same as in the solution of exercise 4.

Hence the required probability is

$$p = \int_{\sqrt{(2rR)}}^{\frac{1}{2}r\sqrt{2}} ds \div \int_{0}^{\frac{1}{2}r\sqrt{2}} ds$$

$$= \int_{\sqrt{(2rR)}}^{\frac{1}{2}r\sqrt{2}} (y^2 + r^2)^{\frac{1}{2}}\,dy \div \int_{0}^{\frac{1}{2}r\sqrt{2}} (y^2 + r^2)^{\frac{1}{2}}\,dy$$

$$= 1 - \frac{2\sqrt{(4R^2 + 2rR)} + 2r\log[\sqrt{(2R)} + \sqrt{(2R + r)}] - r\log r}{r\sqrt{3} + 2r\log(\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{6})}.$$

[*Artemas Martin.*]

81

Find the equation of a curve whose ordinates represent the areas of the triangles in exercise 68. [*R. H. Graves.*]

SOLUTION.

The area of one of the triangles is

$$\frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1) = \frac{1}{8a}\sqrt{(-27\Delta)},$$

where $\Delta =$ discriminant of the cubic,

$$y'^3 + (3a^2 - 4ax)y' - 8a^2y = 0.$$

$$\Delta = 64a^4y^2 + \frac{4}{3}(8a^2 - 4ax)^3$$
$$= 64a^4 . 16a(x - 6a) + \frac{4}{3}(8a^2 - 4ax)^3,$$

by equation to locus, $y^2 = 16a(x - 6a)$. Whence

$$\frac{1}{8a}\sqrt{(-27\Delta)} = 2\sqrt{[a(x - 8a)^2(x + 10a)]};$$
$$\therefore \quad y^2 = 4a(x - 8a)^2(x + 10a)$$

is the required equation. It represents a parabola of the third degree having a loop and a node. The triangles correspond to the ordinates of those points for which $x > 6a$. [*R. H. Graves.*]

83

THE middle point of the segment AB is M. Find the locus of P when PM is a mean proportional between PA and PB. [*L. G. Carpenter.*]

SOLUTION I.

Assuming that AB is a straight line, let

$$AB = c, \ PM = m_c, \ PA = b, \ PB = a.$$

Equating 4 and 43 of Baker's *Formulæ for the Area of a Plane Triangle** we have

$$c = \sqrt{2(a^2 + b^2 - 2m_c^2)}$$
$$= \sqrt{2} . (a - b),$$

since m_c is a mean proportional between a and b. Hence the locus of P is an hyperbola whose foci are A and B; centre, M; and eccentricity, $\sqrt{2}$.

[*Ormond Stone.*]

SOLUTION II.

$$AB^2 = 2PA^2 + 2PB^2 - 4PM^2$$
$$= 2PA^2 + 2PB^2 - 4PA . PB;$$
$$\therefore \sqrt{2} . MB = \pm(PA - PB);$$

therefore the required locus is an equilateral hyperbola.

[*R. H. Graves.*]

SOLUTION III.

Let $MP = r$; then, if x and y be the co-ordinates of P, we have

$$AP^2 = (a + x)^2 + y^2,$$
$$BP^2 = (a - x)^2 + y^2;$$

**Annals of Mathematics*, Vol. I. p. 136, and Vol. II. p. 12.

whence, by the conditions of the problem,

$$r^4 = (x^2 + y^2)^2 = [(a + x)^2 + y^2]\,[(a - x)^2 + y^2];$$
$$\therefore\; x^2 - y^2 = \tfrac{1}{2}a^2,$$

a rectangular hyperbola, semi-major axis $a\sqrt{\tfrac{1}{2}}$. [*Ormond Stone.*]

84

FIND the thinnest frustum that can be cut from a given right circular cone by a plane parallel to the base, subject to the condition, that it may be laid on its slant surface on a horizontal plane without toppling over.

[*W. M. Thornton.*]

SOLUTION I.

Let X be the distance from the vertex of the cone to its centre of gravity, x_1 the distance to the centre of gravity of the cone that is cut off, and x the distance from the vertex to the centre of gravity of the frustum. Let M and m be the masses of the cones, and H and h their heights. For the centre of gravity of the frustum we have

$$\frac{x - X}{X - x_1} = \frac{m}{M - m},$$

or

$$x(M - m) = X(M - m) + (X - x_1)\,m.$$

Now

$$X = \tfrac{3}{4}H, \quad x_1 = \tfrac{3}{4}h;$$

and, since M and m are proportional to H^3 and h^3, we have

$$x(H^3 - h^3) = \tfrac{3}{4}H(H^3 - h^3) + \tfrac{3}{4}h^3(H - h),$$

or

$$x = \tfrac{3}{4}\left(H + \frac{h^3}{H^2 + Hh + h^2}\right).$$

If α be the angle between the axis of the cone and an element of its surface, the required condition gives

$$h = x\cos^2\alpha.$$

We have, therefore,

$$(4 - 3\cos^2\alpha)\,h^3 + (4 - 3\cos^2\alpha)\,Hh^2 + (4 - 3\cos^2\alpha)\,H^2h - 3H^3\cos^2\alpha = 0,$$

a complete cubic for h. The height of the frustum is $H - h$. If we put $H = 1$, we have

$$h^3 + h^2 + h - \frac{3\cos^2\alpha}{4 - 3\cos^2\alpha} = 0.$$

[*Asaph Hall.*]

SOLUTION II.

The line which joins the centre of gravity of the frustum to any point of the upper base must be perpendicular to the slant side; whence it follows that the

distance of this point from the upper base is

$$y = r \cdot \frac{R-r}{H-h} = \frac{Rr}{H} = h \tan^2 a.$$

The masses of the whole cone and of the frustum are as H^3 and $H^3 - h^3$; and the distances of their centres of gravity from that of the upper cone, $\frac{3}{4}(H-h)$ and $\frac{1}{4}h(1 + 4\tan^2 a)$;

$$\therefore \quad (H^3 - h^3) \cdot h(1 + 4\tan^2 a) = 3H^3(H-h);$$

whence, if we divide by $H - h$ and put $H = 1$, we get

$$h^3 + h^2 + h = \frac{3}{1 + 4\tan^2 a}.$$

[*W. M. Thornton.*]

[Mr. Geo. Eastwood sends a nearly equivalent solution.]

85

FIND the locus of the points from which the sum of the squares of the normals to the parabola $y^2 = 2px$ is constant.

SOLUTION.

Let x_1, y_1; x_2, y_2; x_3, y_3; x, y be the co-ordinates of the feet of the normals and their point of concourse. Then y_1, y_2, y_3 are the roots of the cubic in y',

$$y'^3 + 2p(p-x)y' - 2p^2y = 0.$$

(Compare solution to exercise 68).

Then $$\Sigma[(x-x_1)^2 + (y-y_1)^2] = c^2, \text{ a constant,}$$

or $$3(x^2+y^2) - 2x\Sigma x_1 + \Sigma x_1^2 - 2y\Sigma y_1 + \Sigma y_1^2 = c^2,$$

or $$3(x^2+y^2) - \frac{x}{p}\Sigma y_1^2 + \frac{1}{4p^2}\Sigma y_1^4 + \Sigma y_1^2 = c^2,$$

or $$3(x^2+y^2) + \frac{x}{p}4p(p-x) + \frac{8p^2(p-x)^2}{4p^2} - 4p(p-x) = c^2;$$

$$\therefore \quad x^2 + 3y^2 + 4px = 2p^2 + c^2,$$

which shows the required locus to be an ellipse. [*R. H. Graves.*]

91

An equilateral triangle, side $2a$, slides in a plane with one angular point on each of two rectangular axes. Find the locus of the third vertex.

SOLUTION I.

The motion is equivalent to that of a circle rolling inside another of double its diameter. The third vertex, being a carried point, describes an ellipse. De-

scribe a circle about the right triangle whose vertices are the sliding points and the origin, and draw a right line from its centre to the third vertex. Then from the points of intersection of the line and circle draw lines to the origin. These lines, which bisect the angles between the axes, give the positions of the axes of the curve. The semi-axes are equal in magnitude to the sum and difference of the altitude and half the side of the equilateral triangle, i. e. $a(\sqrt{3} \pm 1)$. (See Williamson's *Differential Calculus* [1884], Arts. 285 and 295.)

Remark.—The positions and magnitudes of the axes seem to be evident also from the symmetry of the figure. [*R. H. Graves.*]

SOLUTION II.

Let the base AB have its extremities A, B on Ox, Oy, and the vertex C will have for co-ordinates

$$x = 2a \sin(30^\circ + \varphi) = a(\cos\varphi + \sqrt{3} \cdot \sin\varphi),$$
$$y = 2a \cos(30^\circ - \varphi) = a(\sin\varphi + \sqrt{3} \cdot \cos\varphi);$$

if $AB = 2a$ and $BAO = \varphi$.

Eliminating φ, we get for the locus

$$x^2 - xy\sqrt{3} + y^2 = a^2,$$

an ellipse with its centre at O, and its axes in the bisectrices of xOy, whose lengths are

$$2a\sqrt{(4 \pm \sqrt{12})}.$$ [*Geo. Eastwood.*]

[More generally, for any triangle, if h be the altitude and a, b the segments into which it divides the base, the vertex is

$$x = a\cos\varphi + h\sin\varphi, \quad y = h\cos\varphi + b\sin\varphi;$$

and its locus is the ellipse,

$$(hx - ay)^2 + (hy - bx)^2 = (h^2 - ab)^2.$$ *W. M. T.*]

SOLUTION III.

Let d be the abscissa of A; then

$$y^2 + (x - d)^2 = 4a^2,$$
$$d = 2a\cos\varphi;$$
$$\therefore \quad d = y\sqrt{3} - x,$$

and the equation of the locus is

$$x^2 - xy\sqrt{3} + y^2 = a^2,$$

an ellipse with its centre at the origin, axes inclined at angles of 45° to the co-or-

dinate axes, and semi-axes equal to

$$a\sqrt{(4 \pm \sqrt{12})}.$$ [*Frank Muller.*]

93

FIND the radius of the circle inscribed in the evolute of an ellipse, and the ratio in which each point of contact divides the quadrant on which it lies.

[*R. H. Graves.*]

SOLUTION I.

Let $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ be the equation to the ellipse;

then $$\left(\frac{ax'}{a^2 - b^2}\right)^{\frac{2}{3}} + \left(\frac{by'}{a^2 - b^2}\right)^{\frac{2}{3}} = 1$$

will be the equation to the evolute. Since from the form of the equation the centres of the evolute and the inscribed circle co-incide, the radius of the latter is equal to the minimum value of r'.

The problem thus resolves itself, practically, into the determination of the polar co-ordinates (r'_0, θ'_0) of the points of contact.

$$\xi = \left(\frac{a^2 - b^2}{a}\right)^{\frac{1}{3}} \cos\psi, \quad \eta = \left(\frac{a^2 - b^2}{b}\right)^{\frac{1}{3}} \sin\psi$$

are the co-ordinates of a point on the ellipse whose equation is

$$\left(\frac{a}{a^2 - b^2}\right)^{\frac{2}{3}} \xi^2 + \left(\frac{b}{a^2 - b^2}\right)^{\frac{2}{3}} \eta^2 = 1,$$

and $x' = \xi^3$, $y' = \eta^3$ satisfy the equation to the evolute.

It is evident that a minimum value of r' will correspond to a minimum value of

$$x'^2 + y'^2 = \xi^6 + \eta^6 = \left(\frac{a^2 - b^2}{a}\right)^2 \cos^6\psi + \left(\frac{a^2 - b^2}{b}\right)^2 \sin^6\psi.$$

Differentiating with regard to ψ, this becomes

$$-6\left(\frac{a^2 - b^2}{a}\right)^2 \cos^5\psi \sin\psi + 6\left(\frac{a^2 - b^2}{b}\right)^2 \sin^5\psi \cos\psi;$$

whence dividing by $$-6(a^2 - b^2)^2 \sin\psi \cos\psi,$$

and equating to zero, $$\frac{\cos^4\psi}{a^2} - \frac{\sin^4\psi}{b^2} = 0,$$

or $$\tan^2\psi = \pm\frac{b}{a}.$$

The upper sign corresponds to the values

$$\xi_0 = \pm \left(\frac{a^2 - b^2}{a}\right)^{\frac{1}{3}} \left(\frac{a}{a+b}\right)^{\frac{1}{2}},$$

$$\eta_0 = \pm \left(\frac{a^2 - b^2}{b}\right)^{\frac{1}{3}} \left(\frac{b}{a+b}\right)^{\frac{1}{2}};$$

$$\therefore \quad x'_0 = \xi_0^3 = \pm (a-b) \left(\frac{a}{a+b}\right)^{\frac{1}{2}},$$

$$y'_0 = \eta_0^3 = \pm (a-b) \left(\frac{b}{a+b}\right)^{\frac{1}{2}};$$

$$\therefore \quad r'_0 = a - b,$$

$$\tan \theta'_0 = \pm \sqrt{\frac{b}{a}}.$$

[*Ormond Stone.*]

SOLUTION II.

At Fagnani's point on the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad x^2 = \frac{a^3}{a+b} \quad \text{and} \quad y^2 = \frac{b^3}{a+b}.$$

Substituting in the values of the co-ordinates of the centre of curvature,

$$\alpha = \frac{(a^2 - b^2)\, x^3}{a^4} \quad \text{and} \quad \beta = -\frac{(a^2 - b^2)\, y^3}{b^4},$$

we obtain
$$\alpha = \frac{a^{\frac{1}{2}} (a-b)}{(a+b)^{\frac{1}{2}}} \quad \text{and} \quad \beta = -\frac{b^{\frac{1}{2}} (a-b)}{(a+b)^{\frac{1}{2}}};$$

$$\therefore \quad \alpha^2 + \beta^2 = (a-b)^2.$$

Also the perpendicular from the centre on the normal at Fagnani's point $= a - b$; $\therefore$ the circle $x^2 + y^2 = (a-b)^2$ touches the evolute at the point (α, β). The values of the radii of curvature at the ends of the axes are a^2/b and b^2/a; the radius of curvature at Fagnani's point $= \sqrt{(ab)}$; $\therefore$ the ratio of one segment of the quadrant to the other is

$$\left[\frac{a^2}{b} - \sqrt{(ab)}\right] \left[\sqrt{(ab)} - \frac{b^2}{a}\right] = a^{\frac{3}{2}} : b^{\frac{3}{2}}.$$

[*R. H. Graves.*]

94

O is the centre of the circumscribed circle of *ABC*, and *D, E, F* the middle points of its sides. Show that

$$OD^2 + OE^2 + OF^2 = 2R'(2R' - r'),$$

where R', r' are the radii of the circumscribed and inscribed circles of the triangle of the feet of the altitudes. [*R. D. Bohannan.*]

SOLUTION.

Let A', B', C' be the feet of the altitudes, H their intersection point, G the intersection of the medians, and O' the circumcentre of $A'B'C'$, which circle contains also DEF.

It is well known that the radius of the circumcircle of ABC is $2R'$; whence we see that

$$OD^2 + OE^2 + OF^2 = 12R'^2 - \tfrac{1}{4}(a^2 + b^2 + c^2).$$

Again, since HD is the median of BHC,

$$HB^2 + HC^2 = \tfrac{1}{2}a^2 + 2HD^2;$$

and, in like manner,

$$HD^2 + HA^2 = GA^2 + GD^2 + 2GH^2$$
$$= \tfrac{5}{9}AD^2 + 2GH^2;$$

whence we find $\quad 2HA^2 + HB^2 + HC^2 = \tfrac{10}{9}AD^2 + 4GH^2,$

or, remembering that

$$AD^2 + BE^2 + CF^2 = \tfrac{3}{4}(a^2 + b^2 + c^2),$$

$$HA^2 + HB^2 + HC^2 = \tfrac{1}{3}(a^2 + b^2 + c^2) + 3GH^2,$$

or, since $\quad HA = 2OD, \quad HB = 2OE, \quad HC = 2OF,$

and $\quad GH = \tfrac{4}{3}O'H = \tfrac{4}{3}\rho,$

$$OD^2 + OE^2 + OF^2 = \tfrac{1}{12}(a^2 + b^2 + c^2) + \tfrac{4}{3}\rho^2.$$

From the first and last equations, we get

$$OD^2 + OE^2 + OF^2 = 3R'^2 + \rho^2$$
$$= 4R'^2 - 2R'r',$$

since $\quad \rho^2 = R'^2 - 2R'r'.$

[*W. O. Whitescarver.*]

95

IN exercise 65 let k, l be the lengths of the perpendiculars CD, CE drawn at right angles to CA, CB to meet the base in D, E. Show that

$$\frac{m}{c} = \frac{k}{a} \cdot \frac{l}{b}.$$

[*Ormond Stone.*]

SOLUTION.

Equating two values for the double area of the triangles ABC, CDE, we

have, if h be their common altitude,

$$kl \sin C = hm,$$

$$ab \sin C = hc;$$

whence

$$\frac{kl}{ab} = \frac{m}{c}.$$

[*W. M. Thornton.*]

97

In the triangle ABC two lines drawn from C trisect the side AB. Given c, C, and the angle φ between the trisecants; to solve the triangle.

[*Marcus Baker.*]

SOLUTION.

Refer the figure to the base of the triangle as the x-axis of rectangular co-ordinates, with the origin at its middle point. Let xy be the vertex. The conditions of the problem give

$$x^2 + y^2 - cy \cot C = \tfrac{1}{4}c^2,$$

$$x^2 + y^2 - \tfrac{1}{3}cy \cot C = \tfrac{1}{36}c^2;$$

whence

$$y = \frac{2c}{3 \cot \varphi - 9 \cot C};$$

and the radius vector of the vertex is given by the relation,

$$\rho^2 = \frac{c^2}{12} \cdot \frac{3 \cot \varphi - \cot C}{\cot \varphi - 3 \cot C}.$$

By means of C and c draw the circumcircle, and intersect it with the circle whose radius is ρ. The points thus obtained give the desired solution.

[*W. M. Thornton.*]

101

Express in terms of $\sin^{-1} x$ and $\sin^{-1} y$

$$\tan^{-1} \frac{x + y}{\sqrt{(1 - x^2)} + \sqrt{(1 - y^2)}}.$$

SOLUTION.

If

$$\frac{x + y}{\sqrt{(1 - x^2)} + \sqrt{(1 - y^2)}} = \tan \theta, \quad x = \sin \varphi, \quad y = \sin \psi;$$

we may write

$$\tan \theta = \frac{\sin \varphi + \sin \psi}{\cos \varphi + \cos \psi};$$

whence

$$\sin (\theta - \varphi) + \sin (\theta - \psi) = 0;$$

$$\therefore \quad \theta - \varphi = \psi - \theta + 2n\pi, \quad \text{or} \quad \theta - \psi + (2n + 1)\pi;$$

$$\therefore \quad \theta = \tfrac{1}{2}(\varphi + \psi) + n\pi,$$

except when $\varphi = \psi \pm (2n + 1)\pi$; in which case θ is indeterminate.

[*J. L. Love.*]

102

FIND the relation connecting x, y, z when

$$\cot^{-1}(x + y + z - xyz) = \cot^{-1}x + \cot^{-1}y + \cot^{-1}z.$$

SOLUTION.

Adding $\frac{3}{2}\pi$ to each side, and changing signs, this becomes

$$\tan^{-1}(x + y + z - xyz) = \tan^{-1}x + \tan^{-1}y + \tan^{-1}z,$$

the tangent of which gives*

$$x + y + z - xyz = \frac{x + y + z - xyz}{1 - yz - zx - xy};$$

whence $$x + y + z = xyz, \quad \text{or} \quad yz + zx + xy = 0.$$

[*J. L. Love.*]

103

INTO a conical wine glass a spherical ball is dropped. Find the ratio of the concealed surfaces of the ball and the inside of the glass.

SOLUTION.

In the meridian section of the figure let C be the centre of the sphere, V the vertex of the cone, AB the chord of contact, and H, D the intersections of AB and the sphere with CV.

The conical surface is $$c = \pi \,.\, HB \,.\, VB$$
$$= \pi \,.\, HV \,.\, CD,$$

since, in virtue of the similarity of VBH, VCB,

$$HV : VB = HB : (CB = CD).$$

The spherical surface is $$s = 2\pi \,.\, CD \,.\, HD.$$

Hence the required ratio is $$\frac{s}{c} = \frac{2HD}{HV}.$$ [*O. L. Mathiot.*]

*See Chauvenet's *Trigonometry*, p. 39. eq. (170).